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# Approximating the equations governing rotating fluid motion: a case study based on a quasi-geostrophic model

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Nearly all theoretical work in geophysical fluid dynamics is based on approximate forms of the equations of motion, but the best ground-rules for deriving such approximate forms are not clear. Traditionally, scale analysis and global energy conservation have been the guiding principles. The existence of analogues of Lagrangian potential vorticity conservation has been seen as at least aesthetically desirable, but consequent improvements in practical accuracy have not often been demonstrated. A simple case study is here offered in order to illuminate these issues. The Type 1 quasi-geostrophic model (QG1) is adopted as a reference formulation and several approximations to it are examined. They are formally accurate to zeroth or first-order in a Burger number $B$, but may include some higher-order terms and may imply analogues of global energy conservation and Lagrangian potential vorticity conservation. The approximate forms are all characterized by exclusion of the external Rossby mode, and each is related to a certain geostrophic formulation which is familiar in dynamical oceanography. The various approximations are assessed by examining their behaviour in three test problems which may be treated analytically: finite-amplitude internal Rossby wave propagation, zonal flow stability criteria and linearized internal free waves on baroclinic zonal flows. Two of the problems yield support for the hypothesis that the practical accuracy of an approximation may be improved by including higher-order terms in such a way that a potential vorticity conservation analogue is implied. The validity of this hypothesis in the QG1 case could be further investigated by solving more complicated test problems. Its general applicability cannot of course be claimed on the basis of a single case study; but the results obtained here afford evidence in its favour.

## 1. Introduction

In meteorology and oceanography the use of approximate forms of the equations of rotating fluid motion is widespread. The hydrostatic primitive equations are the basis of many numerical weather-forecasting and general circulation models, and many different geostrophic formulations are used in modelling and theoretical studies. Detailed reviews are given by Phillips (1963), Lorenz (1967), Wiin-Nielsen (1968), Gent & McWilliams (1983) and Eliassen (1984).

The approximate formulations are most easily derived from the complete governing equations by a process of straightforward scale analysis. Terms which are small in the relevant physical context are simply omitted, and the remnant equations constitute the approximate model. An extremely useful by-product of this procedure, at least in the cases that are of interest, is that certain unwanted modes of motion are removed, or 'filtered' from the original equations; the filtering is usually associated with the suppression of a time-derivative term.

Even when it is lent a cloak of rigour and respectability by the use of power series expansion methods, scale analysis seems an uncertain and crude basis for the derivation of approximate models. It is not clear that the omission of small terms is always justifiable; yet, on the other hand, scale estimates of scalar and vector products often exaggerate their magnitudes. Furthermore, the technique is usually applied to components of the momentum equation, or to some differentiated form of it, and hence may depend on the choice of coordinate system. Nevertheless, scale analysis does provide a means of assigning to an approximate model a formal degree of accuracy in terms of some small parameter.

During the late 1950s and early 1960s it became recognized that approximate models are more credible if they imply analogues of the conservation properties of the original equations. Global conservation properties received particular attention: energy, entropy, mass, vorticity and angular momentum budgets were investigated. The approach is exemplified by Lorenz's (1960) classical study. Lorenz derived a hierarchy of geostrophically-balanced models by identifying combinations of approximations in the (vertical) vorticity, (horizontal) divergence and thermodynamic equations which preserve global energy conservation in the absence of forcing and dissipation. (Charney (1962) showed that this hierarchy could also be derived by scale analysis. See also Haltiner 1971 chapter 4.) Apart from offering reliable cures for global energy drifts in numerical integrations, Lorenz's method was useful in providing grounds for choice among a large number of possible approximations of the hydrostatic primitive equations.

The unapproximated equations do of course also possess important Lagrangian conservation properties. Most celebrated of these (Hollmann 1964) is the conservation of Ertel's potential vorticity in frictionless, adiabatic flow. Hoskins (1975) and others have pointed out the desirability of some analogous Lagrangian conservation law being implied by approximate models. In fact, not all of the well-known approximate formulations do possess such analogues. The hydrostatic primitive equations, the Type 1 and Type 2 geostrophic models (Phillips 1963) and the $f$-plane semi-geostrophic equations (Hoskins 1975) all conserve potential vorticity in various modified forms, but the balanced models introduced by Lorenz (1960) in general do not. Neither do the semi-geostrophic equations if $f$-plane approximations are abandoned (McWilliams & Gent 1980). An important consequence of these deficiencies is that the only one of the above approximate formulations that describes potential vorticity conservation in subplanetary scale motion on the sphere is the hydrostatic primitive model – which implies buoyancy/inertia modes as well as geostrophic modes. However, this aspect remained until recently a point of little practical interest since no method of deriving improved geostrophically balanced models was known.

An important advance was made independently by Ripa (1981) and Salmon (1982). They showed that Lagrangian conservation of potential vorticity (as well as global energy conservation) was related to a symmetry property of the Hamiltonian of the original equations. Thus (Salmon 1983) formulations having analogues of both conservation properties can be obtained by approximating the Hamiltonian consistently and deriving the implied equations of motion. These may be unwieldy when written in conventional Eulerian form; but Salmon's analysis shows that the desire to achieve both energy and potential vorticity conservation in approximate models is not merely a pipedream. Salmon (1985) has applied the technique to the problem – noted above – of subplanetary scale geostrophic motion on the sphere.

A key question, which the emergence of the Hamiltonian theory has made even more pressing, is the following: what, if anything, is to be gained by using models that possess analogues of the global and Lagrangian conservation properties of the original equations? From an aesthetic viewpoint, such formulations seem unarguably superior to non-conservative models of the same formal accuracy. In the context of practical numerical modelling, however, the issue is less clear: it is not at all obvious that accuracy of simulation will be improved by using conservative formulations. (The same problem arises also as regards the choice of finite representations for continuous dynamical equations, but that aspect will not be pursued here.)

In the present paper a theoretical case study is reported which attempts to throw some light on the question. A familiar dynamical formulation, which itself possesses good conservation properties, is adopted as a reference model, and various approximate forms of it are posed. Some of these possess energy ($E$) and potential vorticity ($Q$) conservation analogues, some possess the former but not the latter, and some possess neither. By comparing analytically the behaviour of the various approximate forms with the known behaviour of the reference model in a number of problems we are able to offer some evidence that $Q$-conserving formulations are in practice more accurate than other formulations having the same formal accuracy.

The reference model is the familiar Type 1 quasi-geostrophic model, now widely known as QG1. The approximate versions are all long-wave forms from which the external Rossby mode has been filtered by removing a time-derivative term from one of the governing equations. In this respect the approximate models bear the same relation to QG1 as the hydrostatic primitive equations bear to the complete equations and all geostrophic models bear to the hydrostatic primitive equations.

QG1 is amongst the least accurate and sophisticated of the many formulations which attempt to describe nearly-geostrophic motion in the atmosphere and oceans (McWilliams & Gent 1980; Gent & McWilliams 1983, 1984; Williams & Yamagata 1984; Salmon 1985). It is also the most widely studied of these models. However, the position of QG1 in the firmament of geostrophic models is of no concern here. It is chosen as the reference model for the case study because of its good conservation properties and its analytical tractability. The behaviour of the various long-wave forms is to be gauged only in relation to that of QG1.

As is well known, the external Rossby mode is one of the most important features of QG1 and of motion on the horizontal scale of the Rossby radius of deformation. Its exclusion from the long-wave forms to be studied merely implies that their physical applicability is to scales which are much larger than the Rossby radius. The long-wave approximations are in fact refinements of a geostrophic formulation which is used in dynamical oceanography. This formulation is itself closely related to the Type 2 geostrophic model (Phillips 1963). The approximations examined in this paper are therefore extensions of a familiar model; further discussion is given later.

The conservation properties of QG1 and its treatment of long-wave motion are briefly reviewed in §2. In §3 the long-wave approximations are presented; their performance in several test problems is examined in §4. Conclusions and suggestions for further work are contained in §5.

## 2. QG1: the reference model

The adopted QG1 forms of the vorticity and thermodynamic equations are

$$\partial_t \nabla^2\psi + J(\psi, \nabla^2\psi) + \beta\partial_x \psi = f_0 \partial_z w, \tag{2.1}$$

and

$$\partial_t \partial_z \psi + J(\psi, \partial_z \psi) + \frac{N^2 w}{f_0} = 0. \tag{2.2}$$

Here $J$ represents the Jacobian of the bracketed quantities with respect to the zonal and latitude Cartesian coordinates $x$ and $y$:

$$J(a,b) = \partial_x a \partial_y b - \partial_y a \partial_x b.$$

$\partial$ indicates partial differentiation with respect to the subscript independent variable, and $\nabla = \boldsymbol{i}\partial_x + \boldsymbol{j}\partial_y$, $\boldsymbol{i}$ and $\boldsymbol{j}$ being unit vectors in the $x$- and $y$-directions. Geometric height $z$ is the vertical coordinate; $t$ = time; $w$ is the vertical velocity component; $\psi$ is the stream function of the geostrophic flow; $f_0$ and $\beta$ are constant mid-latitude values of the Coriolis parameter $f$ and $\partial_y f$ respectively; and $N = N(z)$ is the reference state buoyancy frequency.

Equations (2.1) and (2.2) describe the free quasi-geostrophic motion of an incompressible fluid on a mid-latitude $\beta$-plane. Derivations and conditions for validity of QG1 are given by Phillips (1963), Pedlosky (1964, 1979), Gill (1982) and others.

Figure 1 shows the domain of the reference model. It is a channel bounded by rigid horizontal surfaces at $z = 0, H$ upon which $w = 0$, and by vertical surfaces at $y = 0, L_y$ upon which the normal geostrophic flow $\partial_x \psi$ vanishes and the zonal average zonal geostrophic flow $\overline{U}^x$ obeys $\partial_t \overline{U}^x = 0$. Cyclic conditions (repeat distance $L_x$) are applied in the zonal direction. Subject to these boundary conditions, (2.1) and (2.2) imply the global conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint E \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0, \tag{2.3}$$

in which the integral extends over $0 \leqslant x \leqslant L_x$, $0 \leqslant y \leqslant L_y$, $0 \leqslant z \leqslant H$, and

$$E = \frac{1}{2}\left\{(\nabla\psi)^2 + \frac{f_0^2}{N^2}(\partial_z \psi)^2\right\}, \tag{2.4}$$

is the energy density per unit mass.

Also implied by (2.1) and (2.2) is the Lagrangian conservation law

$$\partial_t Q + J(\psi, Q) = 0, \tag{2.5}$$

in which

$$Q = \nabla^2\psi + \beta y + \partial_z\left\{\frac{f_0^2}{N^2}\partial_z \psi\right\}. \tag{2.6}$$

Equation (2.5) is analogous to the conservation of Ertel's potential vorticity in inviscid adiabatic flow governed by the unapproximated equations of motion (Bretherton 1966; Green 1970; Kuo 1972). Since (2.5) involves only horizontal advection, the quantity $Q$ itself is not directly analogous to Ertel's potential vorticity; the analogy is between the two conservation laws, not the conserved quantities. However, $Q$ is widely referred to as the quasi-geostrophic potential vorticity, or simply the potential vorticity, and we shall follow this usage here.

Wave solutions of QG1 are easily derived and are well known. It is convenient to

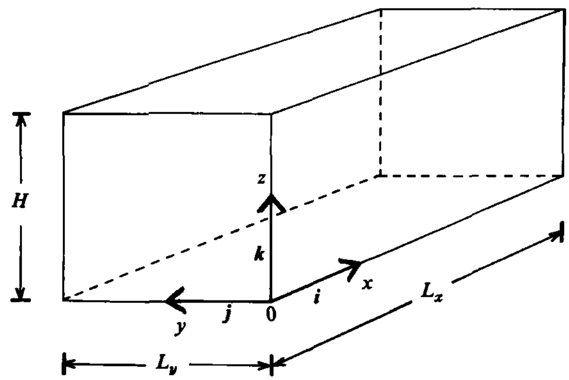


FIGURE 1. The assumed Cartesian domain – a channel of width $L_y$, depth $H$ and zonal repeat distance $L_x$ on a mid-latitude $\beta$-plane. The $0xyz$ coordinate system and the associated unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are also shown. See text for the boundary conditions imposed on the motion.

distinguish external Rossby modes and internal Rossby modes. The former have streamfunction $\psi$ independent of height, and hence $w = 0$ everywhere. The latter have $\psi$ a function of height, with $w = 0$ only at nodes. When the zonal mean flow is uniform (and equal to $U_*$) the external modes have

$$\psi = A_0 \begin{matrix}\sin\\ \cos\end{matrix} k(x - c_0 t) \sin ly - U_* y, \tag{2.7}$$

with
$$c_0 = U_* - \frac{\beta}{k^2 + l^2}, \tag{2.8}$$

and the internal modes have

$$\psi = A_r \begin{matrix}\sin\\ \cos\end{matrix} k(x - c_0 t) \sin ly \cos\left(\frac{r\pi z}{H}\right) - U_* y, \tag{2.9}$$

with
$$c_r = U_* - \frac{\beta}{(k^2 + l^2 + r^2\pi^2 f_0^2/N^2H^2)}. \tag{2.10}$$

Here $A_0, A_r$ are arbitrary constants, and

$$k = \frac{2\pi m}{L_x}, \quad l = \frac{\pi n}{L_y}, \tag{2.11}$$

$m$, $n$ (and $r$) being non-zero integers.

In a certain parameter range the longest external modes have much larger relative phase speeds than the internal modes of the same horizontal scale. From (2.8), (2.10) and (2.11) it follows that $|c_0 - U_*| \gg |c_1 - U_*|$ for $m = n = 1$ if

$$B \equiv \frac{N^2H^2}{f_0^2 L_y^2} \ll 1. \tag{2.12}$$

$B$ is a Burger number based on the channel width $L_y$ (assumed $\lesssim L_x$). The existence of this differentiation of long-wave relative phase speeds is the physical basis of our study. The approximations of QG1 to be introduced in §3 do not describe the fast-moving external modes and are formally applicable to QG1 motion characterized by slow time evolution and large horizontal scale in a domain which permits $B \ll 1$. It is also required that the $\beta$-effect be large enough to allow differentiation between

$|c_0 - U_*|$ and velocity fluctuations in the fluid. If $V$ is an appropriate horizontal velocity amplitude, the condition $R \equiv V/\beta L_y^2 \ll 1$ is sufficient.

According to (2.12), the Burger number $B$ is defined such that its square root is the ratio of the Rossby radius of deformation, $NH/f_0$, to the horizontal scale $L_y$ of the motion. 'Long waves' in the present context are therefore waves whose horizontal scale is substantially greater than $NH/f_0$.

The conditions $B, R \ll 1$ are amply satisfied by planetary scale motion in the Earth's troposphere. With $N^2 = 10^{-4}\ \mathrm{s}^{-2}$, $H = 10^4$ m, $f_0 = 10^{-4}\ \mathrm{s}^{-1}$, $L_y = 10^7$ m, $V = 15\ \mathrm{m\ s^{-1}}$ and $\beta = 1.5\times 10^{-11}\ \mathrm{m^{-1}\ s^{-1}}$, one obtains $B = R = 10^{-2}$. However, approximations to QG1 which are mathematically valid when $B \ll 1$ are not strictly applicable to planetary scales because of the inappropriateness of the $\beta$-plane approximation when $L_y \sim a$ (where $a$ is the planetary radius). For strict physical validity of the long-wave approximations it is required that $NH/f_0 \ll L_y \ll a$. Motion obeying these conditions occurs in the oceans, where $NH/f_0$ is typically of order 50 km. This type of motion (which also obeys $R \ll 1$) is discussed by Gill (1982 p. 531), and by Pedlosky (1979 p. 402).

## 3. Quasi-linear long-wave approximations to QG1

### 3.1. *A twofold decomposition of the stream function*

As might be expected, the external Rossby mode can be removed from QG1 by separating the height-averaged vorticity equation from (2.1) and setting the local time derivative to zero. The chosen decomposition of the stream function $\psi$ is therefore the twofold form

$$\psi = \psi_1 + \psi_2 \tag{3.1}$$

where

$$\psi_1 = \overline{\psi}^z \tag{3.2}$$

and

$$\psi_2 = \psi - \overline{\psi}^z. \tag{3.3}$$

Throughout this paper the overbar is used to indicate averaging over the domain of the associated independent variable or variables (following Green 1970). Thus $\psi_1(=\psi_1(x,y,t))$ is the height average of $\psi$, and $\psi_2(=\psi_2(x,y,z,t))$ is the deviation of $\psi$ from that average. In a customary terminology, $\psi_1$ and $\psi_2$ may be called the barotropic and baroclinic components of the stream function. Separations of this type are familiar from the work of Wiin-Nielsen (1962), Boville (1980) and others.

It is convenient to define

$$J_{ij} = J(\psi_j, \nabla^2\psi_i). \tag{3.4}$$

$J_{ij}$ represents the advection of the relative vorticity $\nabla^2\psi_i$ by the (non-divergent) geostrophic flow $\boldsymbol{v}_j$ which corresponds to $\psi_j$ ($\boldsymbol{v}_j = \boldsymbol{k}\times\nabla\psi_j$, $\boldsymbol{k}$ being unit vertical vector). With each $J_{ij}$ is associated a tracer parameter $n_{ij}$ such that $n_{ij}$ appears always as $n_{ij}J_{ij}$. When $n_{ij} = 1$ the term $J_{ij}$ is retained in its usual form, but when $n_{ij} = 0$ the term $J_{ij}$ is omitted. Tracer parameters are fundamental to all subsequent development in this paper. With the above scheme (2.1) may be separated into equations describing the time evolution of $\nabla^2\psi_i (i = 1, 2)$ as:

$$n_{11}(\partial_t \nabla^2\psi_1 + J_{11}) + n_{22}\overline{J}_{22}^z + \beta\partial_x\psi_1 = 0, \tag{3.5}$$

$$n_{21}(\partial_t \nabla^2\psi_2 + J_{21}) + n_{12}J_{12} + n_{22}(J_{22} - \overline{J}_{22}^z) + \beta\partial_x\psi_2 + \partial_t\partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi_2\right) + J\left(\psi_1 + \psi_2, \partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi_2\right)\right) = 0. \tag{3.6}$$

In writing (3.6) the thermodynamic equation (2.2) has been used to substitute for the vertical velocity $w$. No approximation of (2.2) is contemplated, and so tracer parameters have not been associated with its constituent terms.

The tracer scheme used in (3.5), (3.6) is not the most general possible since $n_{ij}$ is associated with the term $\partial_t \nabla^2 \psi_i$ as well as with $J_{i1}$. This choice has been made in order that the approximate models obtained when $n_{11}$ or $n_{21}$ is set to zero should behave reasonably under zonal Galilean transformation.

Approximate forms of (3.5), (3.6) may be identified unambiguously by a symbolic matrix of the tracer parameters:

$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}, \tag{3.7}$$

in which each $n_{ij} = 0$ or 1. Clearly there are $2^4 - 1 = 15$ possible approximations to QG1 according to the adopted scheme. Of these, 8 have $n_{11} = 0$ and hence do not imply the external Rossby mode (see below). In fact, some of the approximations having $n_{11} = 0$ are not always consistently posed; for if $n_{11} = 0$, (3.5) becomes, in the zonal average,

$$n_{22} \overline{J}^{xz}_{22} = 0. \tag{3.8}$$

Since there is no assurance from (3.6) that $\overline{J}^{xz}_{22} = 0$, (3.8) contradicts (3.6) unless $n_{22} = 0$. All problems involving linearization about a zonal flow obey $\overline{J}^{xz}_{22} = 0$, and it is satisfied in certain other cases too (see §4.1). We shall refer to motion in which $\overline{J}^{xz}_{22} = 0$ as quasi-linear, and will limit attention in this section to such motion. The general case, in which $\overline{J}^{xz}_{22}$ does not necessarily vanish, is discussed further in §5.

### 3.2. *Global energy conservation*

From (3.5), (3.6) and the boundary conditions specified in §2 it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint \left\{ n_{11} (\nabla \psi_1)^2 + n_{21} (\nabla \psi_2)^2 + \frac{f_0^2}{N^2} (\partial_z \psi_2)^2 \right\} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$
$$= 2 \iiint \{ n_{22} \psi_1 \overline{J}^{z}_{22} + n_{21} \psi_2 J_{21} \} \, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z. \tag{3.9}$$

By using the identity

$$\psi_i J_{ij} + \psi_j J_{ii} = \nabla \cdot \{ [\boldsymbol{v}_j \psi_i + \boldsymbol{v}_i \psi_j] \nabla^2 \psi_i \}, \tag{3.10}$$

it is readily shown that the volume integral on the right-hand side of (3.9) vanishes if $n_{22} = n_{21}$. Hence, if this condition is satisfied, the global energy budget (3.9) is analogous to the QG1 form (2.3) whatever the values of $n_{11}, n_{12}$. The tracer matrix for energy-conserving approximations to QG1 is therefore

$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{21} \end{bmatrix}, \tag{3.11}$$

(in which there are three independent elements) and the energy density is given by

$$2E(n_{11}, n_{21}) = n_{11} (\nabla \psi_1)^2 + n_{21} (\nabla \psi_2)^2 + \frac{f_0^2}{N^2} (\partial_z \psi_2)^2. \tag{3.12}$$

Such approximations will be referred to as E-conserving.

### 3.3. *Quasi-geostrophic potential vorticity conservation*

Addition of (3.5) and (3.6) gives

$$(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})\left\{n_{11}\nabla^2\psi_1 + n_{21}\nabla^2\psi_2 + \beta y + \partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi_2\right)\right\}$$
$$= (n_{11}-n_{12})J_{12} + (n_{21}-n_{22})J_{22}, \quad (3.13)$$

in which $\boldsymbol{v} = \boldsymbol{k}\times\boldsymbol{\nabla}\psi = \boldsymbol{v}_1 + \boldsymbol{v}_2$. Thus, if $n_{12} = n_{11}$ and $n_{22} = n_{21}$ there exists an analogue of the QG1 potential vorticity equation (2.5) in terms of

$$Q(n_{11}, n_{21}) = n_{11}\nabla^2\psi_1 + n_{21}\nabla^2\psi_2 + \beta y + \partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi_2\right). \quad (3.14)$$

Such approximations to QG1 will be referred to as $Q$-conserving. They are identified by the tracer matrix

$$\begin{bmatrix} n_{11} & n_{11} \\ n_{21} & n_{21} \end{bmatrix}, \quad (3.15)$$

in which there are but two independent elements.

An important feature of (3.11) and (3.15) is that all $Q$-conserving approximations are also $E$-conserving. This case study cannot therefore illuminate situations in which an approximate model possesses an analogue of potential vorticity conservation but not an analogue of global energy conservation. Some examples of this type are discussed by Gent & McWilliams (1984).

### 3.4. *External and internal Rossby modes in the approximate models*

Consider the problem of small-amplitude waves on a uniform zonal flow $U_*$. Linearization of (3.5), (3.6) in this case (with $N^2$ assumed independent of height) gives

$$n_{11}[\partial_t + U_*\partial_x]\nabla^2\psi_1' + \beta\partial_x\psi_1' = 0, \quad (3.16)$$

$$[\partial_t + U_*\partial_x]\left\{n_{21}\nabla^2\psi_2' + \frac{f_0^2}{N^2}\partial_z^2\psi_2'\right\} + \beta\partial_x\psi_2' = 0, \quad (3.17)$$

(in which the $\psi_i'$ are perturbations). External modes of the form

$$\psi_1' = A_1\,\mathrm{e}^{\mathrm{i}k(x-ct)}\sin ly, \quad \psi_2' = 0, \quad (3.18)$$

are solutions of (3.16), (3.17) if $n_{11} = 1$, and they then have the correct phase speed

$$c = U_* - \frac{\beta}{k^2+l^2}. \quad (3.19)$$

However, if $n_{11} = 0$ it is easily seen that no external mode solutions exist (i.e. $A_1 = 0$ in (3.18)). So, as expected, the external modes are filtered from (3.5), (3.6) by setting $n_{11} = 0$. This result can be demonstrated for finite amplitude waves also (see §4.1). It is well known that external Rossby modes can be removed from QG1 by omitting the local time derivative of vorticity from (2.1) (see, for example, Wiin-Nielsen 1961). The present analysis shows that the same effect can be achieved by treating the height-average form (3.5) alone in this way.

Internal modes having

$$\psi_2' = A_2\,\mathrm{e}^{\mathrm{i}k(x-ct)}\cos\left(\frac{r\pi z}{H}\right)\sin ly, \quad \psi_1' = 0, \quad (3.20)$$

satisfy (3.16) and (3.17) so long as

$$c = U_* - \frac{\beta}{n_{21}(k^2+l^2)+(r^2\pi^2 f_0^2/N^2H^2)}. \tag{3.21}$$

It is notable that $n_{11}$ does not appear in (3.21). So filtering of the external modes can be achieved without simultaneous removal of the internal modes. Further, by retaining $n_{21} = 1$ in (3.6) (and hence in (3.17)) the QG1 phase speeds of the internal modes can be preserved unchanged even if the external modes are filtered by setting $n_{11} = 0$ in (3.5). This result (which can be generalized to finite amplitude, see §4.1) is not profound, but it is thought to be new.

### 3.5. *Two Q-conserving approximations to QG*1

When external modes are removed by setting $n_{11} = 0$ in (3.5), two possible $Q$-conserving formulations may be obtained (depending on the value of $n_{21}$). It is helpful to consider these formulations, and in particular their relation to other published dynamical models. The $Q$-conserving model having $n_{11} = a$, $n_{21} = b$ will be referred to as 'model *ab*'. Thus model 11 is QG1 itself.

Model 00 is the most drastic approximation within our adopted scheme (since all four $n_{ij}$ are zero). Equation (3.5) reduces to

$$\beta \partial_x \psi_1 = 0, \tag{3.22}$$

and (3.6) becomes

$$\partial_t \partial_z\left(\frac{f_0^2}{N^2}\partial_z \psi_2\right) + J\left(\psi_1+\psi_2, \partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi_2\right)\right) + \beta\partial_x\psi_2 = 0. \tag{3.23}$$

The energy density and potential vorticity analogues are given by the simple forms

$$2E(0,0) = \frac{f_0^2}{N^2}(\partial_z \psi_2)^2, \tag{3.24}$$

and

$$Q(0,0) = \beta y + \partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi_2\right), \tag{3.25}$$

(see (3.12) and (3.14)).

Model 00 is identical with the formulation used in dynamical oceanography to describe motion on a scale intermediate between the Rossby deformation radius and the gyre scale (Gill 1982; Pedlosky 1979). Also, in that it completely neglects relative vorticity advection and retains time dependence only through the thermodynamic equation (2.2), model 00 is similar to Phillips' (1963) 'Type 2' model of planetary geostrophic motion. However, the Type 2 model (as defined by Phillips) involves use of spherical geometry, full latitude variation of the Coriolis parameter and special treatment of horizontal boundary conditions. See also Bates (1977) and Lynch (1979).

The internal modes (3.20) allowed in model 00 have phase speed

$$c_r = U_* - \frac{\beta N^2 H^2}{r^2\pi^2 f_0^2}. \tag{3.26}$$

This differs from the QG1 value (see (2.10) and (3.21)) and (as noted by Welander 1961, Wiin-Nielsen 1961 and others) is independent of the horizontal wavenumber

$(k^2+l^2)^{\frac{1}{2}}$. Model 01, in contrast, has internal modes with the correct QG1 phase speeds (since $n_{21}=1$); its governing equations are

$$\bar{J}^z_{22}+\beta\partial_x\psi_1=0, \tag{3.27}$$

$$\partial_t\nabla^2\psi_2+J_{21}+J_{22}-\bar{J}^z_{22}+\beta\partial_x\psi_2+\partial_t\partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi_2\right)+J\left(\psi_1+\psi_2,\partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi_2\right)\right)=0, \tag{3.28}$$

and the energy density and potential vorticity analogues are given by

$$2E(0,1)=(\nabla\psi_2)^2+\frac{f_0^2}{N^2}(\partial_z\psi_2)^2 \tag{3.29}$$

$$Q(0,1)=\nabla^2\psi_2+\beta y+\partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi_2\right). \tag{3.30}$$

Wiin-Nielsen (1961) introduced a modification of the Type 2 model in which the local time derivative of the vorticity was omitted but the advection of relative vorticity retained. Thus external Rossby modes were still removed, although an apparently less drastic approximation had been made. Because it does not preserve invariance of the equations under zonal Galilean transformation this approximation has no counterpart in the scheme used here.

There are two approximate models which are $E$-conserving but not $Q$-conserving. One has $n_{11}=n_{21}=n_{22}=0$, $n_{12}=1$; it will be referred to as model $E00$. The other has $n_{12}=n_{21}=n_{22}=1$, $n_{11}=0$ and will be designated model $E01$. Since it neglects only one term from the full QG1 form, model $E01$ seems the least severe of the approximations which do not imply the external mode.

### 3.6. *Scale analysis in terms of the Burger number*

Although the preliminary discussion at the end of §2 involved a scale consideration of external and internal mode phase speeds (and the importance of this should not be overlooked), the development of approximations to QG1 given in this section so far has not depended on detailed scale analysis. Rather, the approach has been to identify a range of possible approximations, and then to examine the conservation and wave propagation properties of each. This is considered to be a suitable emphasis. However, detailed scale analysis cannot be dispensed with completely because it enables a formal degree of accuracy to be ascribed to each approximate model. It is also of interest to find whether any of the $Q$- or $E$-conserving approximate models appear as readily through scale analysis as they do through examination of conservation properties. Consider (3.5) and (3.6). It is natural to scale $x$ and $y$ by $L_y$, $t$ by $(L_y/V)$ (where $V$ is a typical horizontal velocity amplitude) and $\psi_2$ by $VL_y$. The scaling factor for $\psi_1$ is specified as $\lambda VL_y$ where $\lambda$ is a non-dimensional parameter to be determined later. Other non-dimensional numbers which arise are a Burger number $B$ and a Rossby number $R$ defined by

$$\left.\begin{aligned} B&=\frac{N^2H^2}{f_0^2L_y^2},\\ R&=\frac{V}{\beta L_y^2},\end{aligned}\right\} \tag{3.31}$$

(cf. §2).

Equations (3.5) and (3.6) may then be written in non-dimensional form as

$$B\lambda n_{11}[\partial_t \nabla^2\psi_1 + \lambda J_{11}] + Bn_{22}\overline{J}^z_{22} + BR^{-1}\lambda\partial_x\psi_1 = 0, \tag{3.32}$$

$$B\{n_{21}[\partial_t \nabla^2\psi_2 + \lambda J_{21}] + \lambda n_{12}J_{12} + n_{22}[J_{22} - \overline{J}^z_{22}]\} + BR^{-1}\partial_x\psi_2 + \partial_t\partial_z^2\psi_2 + J(\lambda\psi_1 + \psi_2, \partial_z^2\psi_2) = 0. \tag{3.33}$$

In deriving these forms, $N$ has been assumed independent of height. $\psi_1$ and $\psi_2$ in (3.32) and (3.33) are non-dimensional, as are the operators $\partial_t, \partial_x$ and $\nabla^2$, their dimensional counterparts having been scaled in the obvious way, given the adopted scalings of the independent variables. Both (3.32) and (3.33) have been multiplied through by $B$ in the derivation from (3.5) and (3.6).

To order $B^0$, (3.33) reduces to a trivial form unless $R$ is taken to be of order $B$. With this relation (which is consistent with the discussion given in §2), (3.33) becomes equivalent, to order $B^0$, to (3.23) of model 00 (so long as $\lambda$ is of order unity or less). However, $R = O(B)$ must be applied also in (3.32), which then contains a term in $\partial_x\psi_1$ that is of order $\lambda B^0$. Taking $\lambda = 1$ then implies that (3.32) reduces to

$$\partial_x\psi_1 = 0, \tag{3.34}$$

at leading order ($B^0$). This is consistent with (3.22), of model 00. But (3.34) suggests that the leading-order balance in (3.32) is trivial, and hence that $\lambda = 1$ was an inappropriate setting (at least for the scaling of $[\psi_1 - \overline{\psi}^x_1]$). The structure of (3.32) evidently requires $\lambda = B$. Such a scaling was used by Pedlosky (1977) in a study of amplitude vacillation in a 2-level, $\beta$-plane model. Applying $\lambda = B$ (which still implies (3.34) to order $B^0$) as the appropriate setting, (3.32) gives

$$\overline{J}^z_{22} + BR^{-1}\partial_x\psi_1 = 0, \tag{3.35}$$

at order $B$. This is equivalent to (3.27) of model 01. If $n_{11} = 1$ (in which case the external Rossby mode is retained) terms of order $B^2$ and $B^3$ are retained in (3.32).

Applying $\lambda = B$ to (3.33) shows that all terms of order $B$ are retained if $n_{21} = n_{22} = 1$ and some terms of order $B^2$ if $n_{12}$ or $n_{21} = 1$.

All approximations having $n_{21} = n_{22} = 1$ are formally accurate to order $B$ but all others only to order unity. Thus models 00 and $E$00 are accurate to order unity (but retain some terms of order $B$). Models 01 and $E$01 are accurate to order $B$ (but retain some terms of order $B^2$). There is no non-conserving model having $n_{11} = 0$ which is accurate to order $B$.

It is interesting that none of the conserving models 00, $E$00, 01 and $E$01 are obvious approximations on the basis of scale analysis. (This arises mainly because of our requirement that all approximations should be invariant under zonal Galilean transformation.) The retention of terms of a higher order than the formal accuracy of an approximation is a feature of various published geostrophic formulations. For example, in the semi-geostrophic model (Hoskins 1975), geostrophic advection of ageostrophic momentum is neglected, but ageostrophic advection of geostrophic momentum is retained (see the discussion following Fjørtoft 1962). It is also characteristic of the global geostrophic models proposed by Salmon (1983, 1985). The hope in each case must be that the retention of some higher-order terms, but not all, leads to improved accuracy in practice because conservation properties are improved, either because a conservation analogue exists where otherwise there would be none, or because the conservation analogue is itself rendered more realistic.

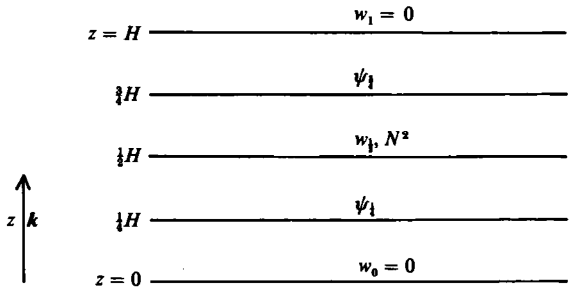


FIGURE 2. Arrangement of variables and constants in the two-level model.

### 3.7. *Two-level models*

The analysis given in §§3.1–3.6 for the QG1 model with continuous vertical structure can easily be applied to a two-level model based on (2.1) and (2.2) (subject to the usual boundary conditions). Figure 2 shows the assumed arrangement of levels and variables. By writing (2.1) and (2.2) in standard finite-difference forms and introducing tracer parameters $n_{ij}$ as before, the two-level analogues of (3.5), (3.6) may be obtained as:

$$n_{11}[\partial_t \nabla^2 \psi_1 + J_{11}] + n_{22} J_{22} + \beta \partial_x \psi_1 = 0, \tag{3.36}$$

$$n_{21}[\partial_t \nabla^2 \psi_2 + J_{21}] + n_{12} J_{12} + \beta \partial_x \psi_2 = \mu^2[\partial_t \psi_2 + J(\psi_1, \psi_2)]. \tag{3.37}$$

Here $\psi_1$ and $\psi_2$ are the barotropic and baroclinic stream functions defined by

$$\left.\begin{aligned} \psi_1 &= \tfrac{1}{2}(\psi_{\frac{3}{4}} + \psi_{\frac{1}{4}}), \\ \psi_2 &= \tfrac{1}{2}(\psi_{\frac{3}{4}} - \psi_{\frac{1}{4}}), \end{aligned}\right\} \tag{3.38}$$

in analogy with the continuous QG1 case, and $\mu^2 = 8f_0^2/N^2H^2$ ($N^2$ being evaluated at $z = \frac{1}{2}H$). Equations (3.36), (3.37) are slightly simpler than (3.5), (3.6) because the Jacobian terms in the two-level model are identically equal to their height averages.

It is readily shown that the tracer matrix $\{n_{ij}\}$ of (3.36), (3.37) behaves exactly as for the continuous model. Thus (3.7) reduces to (3.11) for $E$-conserving approximations, and to (3.15) for approximations that are also $Q$-conserving. The external mode is removed by setting $n_{11} = 0$, and the phase speed of the single internal mode preserved by retaining $n_{21} = 1$ (but is altered if $n_{21} = 0$). Fully conserving approximate models 00 and 01 and energy conserving models $E$00 and $E$01 may be defined in analogy to the continuous case. Finally, it can be shown that the number of possible approximations, the requirement of quasi-linearity ($\overline{J}^x_{22} = 0$) if $n_{22} = 1$ and $n_{11} = 0$, and the suggestions of scale analysis are the same as before.

## 4. Behaviour of the quasi-linear approximations in specific problems

### 4.1. *Finite-amplitude Rossby waves and zonal mean flows*

The QG1 potential vorticity equation (2.5) possesses finite-amplitude solutions having

$$\partial_t = -c\partial_x, \quad Q = F(\psi + cy), \tag{4.1}$$

where $c$ is a real constant and $F$ a single-valued function. A particular case of (4.1), which arises when $F(\ ) = -\kappa^2(\ )$, is the family of Rossby waves and zonal flows having the same total wavenumber $\kappa$; a uniform zonal flow $U_*$ is also permitted (Kuo 1959, 1973; Mitchell & Derome 1983). Assuming for simplicity that $N^2$ is independent of height, the solutions take the form

$$\psi = -U_* y + \sum a_{klr} \,{}^{\sin}_{\cos}\, k(x-ct) \,{}^{\sin}_{\cos}\, ly \cos\left(\frac{r\pi z}{H}\right), \tag{4.2}$$

in which the $a_{klr}$ are arbitrary real constants, $r = 0, 1, \ldots$, and the sum extends only over harmonic elements for which

$$k^2 + l^2 + \frac{r^2\pi^2 f_0^2}{N^2 H^2} = \kappa^2, \tag{4.3}$$

takes the same value. The phase speed $c$ is given by the familiar relation

$$c = U_* - \frac{\beta}{\kappa^2}. \tag{4.4}$$

The vertical structure functions in (4.2) have been chosen so that the boundary conditions $w = 0$ at $z = 0$, $H$ are satisfied. If the horizontal boundary conditions adopted in §2 are also applied, the number of harmonic components which can contribute to (4.2) is severely limited, often to two components. However, if the domain scale ratio $(L_x/L_y)$ and the Burger number $B$ take certain special values, then several different components may be included in (4.2). An example (of many possibilities) is the form

$$\begin{aligned}\psi = -U_* y + a \,{}^{\sin}_{\cos}\left(\frac{(12)^{\frac{1}{2}}\pi y}{L_y}\right) + b \,{}^{\sin}_{\cos}\left(\frac{2\pi y}{L_y}\right) \cos\left(\frac{\pi z}{H}\right) \\ + c \,{}^{\sin}_{\cos}\left[\frac{2\pi(x-ct)}{L_x}\right] \sin\left(\frac{3\pi y}{L_y}\right) + d \,{}^{\sin}_{\cos}\left[\frac{2\pi(x-ct)}{L_x}\right] \sin\left(\frac{\pi y}{L_y}\right) \cos\left(\frac{\pi z}{H}\right),\end{aligned} \tag{4.5}$$

whose harmonic components each have the same total wavenumber if $B = \frac{1}{8}$ and $L_x/L_y = 2/\sqrt{3}$ ($a$, $b$, $c$ and $d$ are arbitrary real constants). This finite-amplitude solution of the inviscid, adiabatic QG1 model is a meagre generalization of one of those used by White (1986*b*) to illustrate certain consequences of wave/mean flow non-interaction. Given that such multi-component solutions of QG1 do exist even when the usual boundary conditions in $x$ and $y$ are imposed, a good test of the various approximate models introduced in §3 is their treatment of solutions of the form of (4.2).

Cases in which $U_* = 0$ may be considered without loss of generality. The solutions (4.2) are readily decomposed into barotropic ($\psi_1$) and baroclinic ($\psi_2$) components, and application of (4.3) shows that the quasi-linearity condition, $\overline{J_{22}}^{xz} = 0$, is obeyed. Further, (3.5) is satisfied if

$$\partial_x \psi_1(\beta + n_{11}\kappa^2 c) = 0. \tag{4.6}$$

Approximations having $n_{11} = 0$ must therefore have $\partial_x \psi_1 = 0$. Thus external Rossby modes are disallowed if $n_{11} = 0$, although a barotropic harmonic zonal flow $U_0 = -\partial_y \overline{\psi_1}^x$ is permitted.

Consideration of (3.6) leads to the more complicated condition

$$\begin{aligned}\partial_x \psi_2\{\beta + n_{21}\kappa^2 c + (n_{12} - n_{21})\kappa^2 U_0\} \\ + (n_{21} - 1)(c - U_0)\frac{f_0^2}{N^2}\partial_x \partial_z^2 \psi_2 + (1 - n_{22})\frac{f_0^2}{N^2} J(\psi_2, \partial_z^2 \psi_2) = 0,\end{aligned} \tag{4.7}$$

| $n_{12}$ | $n_{21}$ | $n_{22}$ | Remarks | Order of formal accuracy | Allowed components | Phase speed |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | Model $E01$ | $B$ | $U_0, \psi_2$ | Correct |
| 0 | 1 | 1 | Model 01 | $B$ | $\psi_2$ | Correct |
| 1 | 1 | 0 | Non-conserving | 1 | $U_0, \psi_2[SM]$ | Correct |
| 0 | 1 | 0 | Non-conserving | 1 | $\psi_2[SM]$ | Correct |
| 1 | 0 | 1 | Non-conserving | 1 | $\psi_2[SM]$ | Approx. |
| 0 | 0 | 1 | Non-conserving | 1 | $\psi_2[SM]$ | Approx. |
| 1 | 0 | 0 | Model $E00$ | 1 | $\psi_2[SM]$ | Approx. |
| 0 | 0 | 0 | Model 00 | 1 | $\psi_2[SM]$ | Approx. |

TABLE 1. Treatment of finite-amplitude internal Rossby modes by 8 approximate forms of QG1

*Notes.* (*a*) $U_0$ is a barotropic zonal flow element. (*b*) $\psi_2$ is a superposition of internal Rossby modes and baroclinic zonal flows each having the same total wavenumber. (*c*) $\psi_2[SM]$ indicates either a single internal Rossby mode (or baroclinic zonal flow) or a superposition of these elements having the same vertical wavenumber (as well as the same total wavenumber). (*d*) The correct phase speed is that given by (4.4); the approximate phase speed is that given by (3.26).

in the case $n_{11} = 0$. Equation (4.7) involves the tracers $n_{12}$, $n_{21}$ and $n_{22}$. Table 1 summarizes the restrictions on the solutions (4.2) which are imposed by (4.7) in each of the eight possible tracer combinations. Since external modes are disallowed when $n_{11} = 0$, the best that can be achieved is retention of the barotropic zonal flow $U_0$ and all internal modes (including baroclinic zonal flows) with the phase speed (4.4) preserved. Table 1 is arranged as a 'league table' of approximations. In the case $n_{12} = n_{21} = n_{22} = 1$, which is the $E$-conserving model $E01$, the best conceivable behaviour is achieved. The $Q$-conserving model 01 runs a very close second: baroclinic components are allowed without restriction, and the phase speed (4.4) is preserved, but the barotropic zonal flow $U_0$ is not allowed. All other approximations (which are formally accurate only to order unity) fall short in more serious ways. Thus, as table 1 shows, phase speeds are altered in some cases, and $\psi_2$ is restricted to consist of only a single component or of components having the same vertical wavenumber (as well as the same total wavenumber).

Since model $E01$ performs marginally better than model 01, $Q$-conservation does not lead in this problem to better results for order $B$ approximations. Of the six formulations which are formally accurate only to order unity, models 00 and $E00$ are amongst the four giving the worst behaviour: better results are given by two non-conserving approximations. However, all the order-unity approximations have serious deficiencies in their performance (see table 1), and models 00 and $E00$ are amongst the most drastic approximations as regards the number of terms omitted from the QG1 forms.

### 4.2. *Zonal flow stability criteria*

Zonal flow stability problems can be investigated via (3.5) and (3.6) by resolving both mean flow $\overline{U}^x$ and perturbation $\psi'$ into their barotropic and baroclinic elements and linearizing in the usual way. With

$$\overline{U}^x(y,z) = \overline{U}_1^x + \overline{U}_2^x = -\partial_y \overline{\psi}_1^x - \partial_y \overline{\psi}_2^x,$$

and

$$\psi' = \psi_1' + \psi_2',$$

(3.5) and (3.6) give

$$(\partial_t + \overline{U}^2 \partial_x) q' + \partial_y Q \partial_x \psi' = \epsilon. \tag{4.8}$$

Here the perturbation potential vorticity $q'$ and mean potential vorticity gradient $\partial_y Q$ are given by

$$q' = n_{11} \nabla^2 \psi_1' + n_{21} \nabla^2 \psi_2' + \partial_z \left( \frac{f_0^2}{N^2} \partial_z \psi_2' \right), \tag{4.9}$$

$$\partial_y Q = \beta - n_{11} \partial_y^2 \overline{U}_1^x - n_{21} \partial_y^2 \overline{U}_2^x - \partial_z \left( \frac{f_0^2}{N^2} \partial_z \overline{U}_2^x \right), \tag{4.10}$$

and the quantity $\epsilon$ takes the form

$$\epsilon = (n_{11} - n_{12}) \{ \overline{U}_2^x \partial_x \nabla^2 \psi_1' - \partial_y^2 \overline{U}_1^x \partial_x \psi_2' \} + (n_{21} - n_{22}) \{ \overline{U}_2^x \partial_x \nabla^2 \psi_2' - \partial_y^2 \overline{U}_2^x \partial_x \psi_2' \}. \tag{4.11}$$

In the QG1 problem $n_{ij} = 1$; $q'$ and $\partial_y Q$ then reduce to their usual forms, and $\epsilon$ vanishes. $\epsilon$ also vanishes for all $Q$-conserving approximations to QG1, but not for $E$-conserving or non-conserving forms.

Stability criteria can be derived from (4.8) by applying familiar manipulations. Consider first the cast of stability to wave mode solutions of the form

$$\left. \begin{aligned} \psi_1' &= F_1(y) \, e^{ik(x-ct)}, \\ \psi_2' &= F_2(y,z) \, e^{ik(x-ct)}. \end{aligned} \right\} \tag{4.12}$$

With $\hat{\epsilon}$ defined such that

$$\epsilon = \hat{\epsilon} \, e^{ik(x-ct)} \tag{4.13}$$

it then follows from (4.8)–(4.10) and the boundary conditions

$$\left. \begin{aligned} F_i(y=0) = F_i(y=L_y) &= 0 \quad (i = 1,2), \\ (\partial_t + \overline{U}^x \partial_x) \partial_z \psi_2' - \partial_z \overline{U}^x \partial_x \psi' &= 0 \quad \text{at } z = 0, H, \end{aligned} \right\} \tag{4.14}$$

that

$$\begin{aligned} &\iint \left\{ n_{11} [|\partial_y F_1|^2 + k^2 |F_1|^2] + n_{21} [|\partial_y F_2|^2 + k^2 |F_2|^2] + \frac{f_0^2}{N^2} |\partial_z F_2|^2 \right\} \mathrm{d}y \, \mathrm{d}z \\ &\quad = \iint \frac{(\overline{U}^x - c^*)}{|\overline{U}^x - c|^2} \partial_y Q |F_1 + F_2|^2 \, \mathrm{d}y \, \mathrm{d}z + \int \left[ \frac{f_0^2}{N^2} \frac{(\overline{U}^x - c^*)}{|\overline{U}^x - c|^2} \partial_z \overline{U}^x |F_1 + F_2|^2 \right]_0^H \mathrm{d}y \\ &\qquad - \iint (F_1^* + F_2^*) \frac{(\overline{U}^x - c^*)}{|\overline{U}^x - c|^2} \hat{\epsilon} \, \mathrm{d}y \, \mathrm{d}z. \end{aligned} \tag{4.15}$$

By considering the real and imaginary parts of (4.15), necessary conditions for instability to wave modes may be obtained for the $Q$-conserving models ($\hat{\epsilon} = 0$). These conditions are similar to those obtained for QG1 by Pedlosky (1964), but here apply also to models 00 and 01 (in terms of the appropriate potential vorticity gradients as defined by (4.10)). But if $\hat{\epsilon} \neq 0$ ($E$-conserving and non-conserving forms) no useful simplification occurs in (4.15), and no stability criterion can be deduced.

For general non-axisymmetric perturbations it follows from (4.8)–(4.10) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\left\{n_{11}(\nabla\psi_1')^2+n_{21}(\nabla\psi_2')^2+\frac{f_0^2}{N^2}(\partial_z\psi_2')^2-\frac{\overline{U}^x(q')^2}{\partial_y Q}\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}\iint\left[\frac{f_0^2\,\overline{U}^x}{N^2\,\partial_z\,\overline{U}^x}(\partial_z\psi_2')^2\right]_0^H\mathrm{d}x\,\mathrm{d}y-2\iiint\left\{\frac{\overline{U}^x q'}{\partial_y Q}+\psi'\right\}\epsilon\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \quad (4.16)$$

Sufficient conditions for stability to such perturbations can be derived from (4.16) in the case of $Q$-conserving models ($\epsilon = 0$). These conditions involve the potential vorticity gradients defined by (4.10) and are similar to the conditions obtainable for QG1 (see Blumen 1978). For all other approximations (including $E$-conserving forms) $\epsilon \neq 0$, and usual stability criteria are not forthcoming.

These results provide evidence that, in some problems at least, $Q$-conserving approximations are superior to $E$-conserving and non-conserving approximations of the same formal accuracy. Further evidence could be sought by solving some stability problems explicitly. In cases which are of interest, however, the analytical complications are considerable; such problems are set aside for future work. The two-level model provides a more tractable basis for simple analytical study.

### 4.3. *Linearized free waves in two-level models*

Consider the case of free waves on the baroclinic zonal flow $\overline{U}_{\frac{1}{4}}^x = U_*-\Delta U$, $\overline{U}_{\frac{3}{4}}^x = U_*+\Delta U$ (see figure 2). The components of the total stream function are

$$\left.\begin{aligned}\psi_1 &= -U_*y+\psi_1',\\ \psi_2 &= -\Delta Uy+\psi_2',\end{aligned}\right\} \quad (4.17)$$

in which the $\psi_j'$ are small perturbations. The linearized forms of (3.36), (3.37) have non-trivial wave mode solutions of the form

$$\psi_j' = A_j\,\mathrm{e}^{\mathrm{i}k(x-ct)}\sin ly, \quad (4.18)$$

so long as

$$(\gamma+n_{11}p^2X)(\gamma+X[8+n_{21}p^2])+n_{22}p^2(8-n_{12}p^2) = 0. \quad (4.19)$$

In (4.19)

$$p^2 = \frac{(k^2+l^2)\,N^2H^2}{f_0^2},\quad \gamma = \frac{\beta N^2H^2}{f_0^2\Delta U},\quad X = \frac{(c-U_*)}{\Delta U}. \quad (4.20)$$

Thus $p$, $\gamma$ and $X$ are respectively non-dimensional wavenumber, $\beta$-parameter and complex phase speed. In the case of QG1 ($n_{ij} = 1$), (4.19) is quadratic in $X$, and for small values of the wavenumber $p$ one of the solutions is the two-level version of an internal Rossby mode on a baroclinic zonal mean flow. This solution has

$$X = X_+ = -\tfrac{1}{8}\gamma\left[1+\tfrac{1}{8}p^2\left(\frac{64}{\gamma^2}-1\right)-\tfrac{1}{64}p^4\left(\frac{64}{\gamma^2}-1\right)+\tfrac{1}{512}p^6\left(\frac{4096}{\gamma^2}-1\right)+O(p^8)\right]. \quad (4.21)$$

The other QG1 solution of (4.19) is the two-level version of an external Rossby mode on a baroclinic zonal mean flow. When $n_{11} = 0$ this solution ($X_-$) is disallowed, and (4.19) reduces to a linear equation:

$$X = -\frac{n_{22}p^2(8-n_{12}p^2)+\gamma^2}{\gamma(8+n_{21}p^2)}. \quad (4.22)$$

According to (4.22), $X$ is always real. Thus removal of the external Rossby mode from the two-level QG1 model simultaneously annihilates baroclinic instability. (An

| $n_{12}$ | $n_{21}$ | $n_{22}$ | Remarks | Order of formal accuracy | Accuracy of phase speed $X$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | Model 01 | $B$ | $O(p^4)$ |
| 1 | 1 | 1 | Model $E$01 | $B$ | $O(p^2)$ |
| 1 | 1 | 0 | Non-conserving | 1 | $O(1)$ |
| 0 | 1 | 0 | Non-conserving | 1 | $O(1)$ |
| 1 | 0 | 1 | Non-conserving | 1 | $O(1)$ |
| 0 | 0 | 1 | Non-conserving | 1 | $O(1)$ |
| 1 | 0 | 0 | Model $E$00 | 1 | $O(1)$ |
| 0 | 0 | 0 | Model 00 | 1 | $O(1)$ |

TABLE 2. Treatment of phase speeds of small-amplitude internal Rossby modes on a simple two-level baroclinic flow by 8 approximate forms of QG1

analogous result does not hold for the continuous vertical structure model (3.5), (3.6): the long wave instabilities described by Green (1960) are still present. See, for example, Lynch 1979.)

However, our concern here is not with baroclinic instability but with the extent to which, in the various approximations, (4.22) reproduces the $X_+$ solution (4.21) of the QG1 problem. Results are summarized in table 2, which is again drawn up in 'league table' form. Clear winner is model 01, the $Q$-conserving form having $n_{12} = 0$ and $n_{21} = n_{22} = 1$. In this case

$$X = -\frac{\gamma^2+8p^2}{\gamma(8+p^2)} = -\tfrac{1}{8}\gamma\left[1+\tfrac{1}{8}p^2\left(\frac{64}{\gamma^2}-1\right)-\tfrac{1}{64}p^4\left(\frac{64}{\gamma^2}-1\right)+\tfrac{1}{512}p^6\left(\frac{64}{\gamma^2}-1\right)+O(p^8)\right], \quad (4.23)$$

which reproduces (4.21) to order $p^4$. Runner up is model $E$01 ($n_{12} = n_{21} = n_{22} = 1$). This gives

$$X = -\frac{\gamma^2+p^2(8-p^2)}{\gamma(8+p^2)} = -\tfrac{1}{8}\gamma\left[1+\tfrac{1}{8}p^2\left(\frac{64}{\gamma^2}-1\right)-\tfrac{1}{64}p^4\left(\frac{128}{\gamma^2}-1\right)+O(p^6)\right], \quad (4.24)$$

which follows (4.21) to order $p^2$. All other possible combinations of $n_{12}$, $n_{21}$ and $n_{22}$ give $X_+$ correct to order 1 only. Thus the $Q$-conserving model 01 is to be preferred to the $E$-conserving model $E$01 for the accuracy it gives in this problem; and models 01 and $E$01 are both to be preferred to all models having lower formal accuracy. Examination of the amplitude ratio $A_1/A_2$ in the various cases gives less clear-cut results. The order-$B$ models 01 and $E$01 give the ratio correct to order $p^2$, but so also do the two non-conserving, order unity approximations having $n_{22} = 1$. All other approximations give $A_1 = 0$.

Other problems which can be easily examined in the two-level formulation include those of steady-state linearized responses to topographic and diabatic forcing. Rather surprisingly, however, no clear discrimination between the various conserving and non-conserving approximate models emerges in these steady-state cases. The two order-$B$ (conserving) models perform equally well, and in all respects at least as well as the order-unity models. The two order-unity, conserving models (00 and $E$00) are inferior to certain non-conserving models in some respects, but no single model having the same formal accuracy performs uniformly better. Details are given in White (1986*a*).

## 5. Discussion

In the present state of knowledge, the relative merits of various conserving and non-conserving models in geophysical fluid dynamics can be firmly established only by conducting extensive series of numerical integrations and intercomparisons: general theoretical results are, as yet, lacking. This paper has described a case study, whose results may suggest some wider conclusions. Of course, by its very nature a case study cannot give universal results, but useful evidence for or against general hypotheses may accrue. The familiar QG1 model has been taken as a reference formulation, and the behaviour of various long-wave approximations examined analytically.

The long-wave approximations to QG1 may be categorized according to their formal accuracy in terms of the Burger number $B$ as well as by their conservation properties. Of approximations having order-$B$ formal accuracy, a $Q$-conserving form has been found to give better results than an $E$-conserving form in two of the problems studied. The $E$-conserving form gives a marginally better performance than the $Q$-conserving form in the remaining problem. Within the adopted analytical framework there exists no non-conserving approximation that is accurate to order $B$, and hence it is not possible in this case to discriminate between conserving and non-conserving models. Approximations which are formally accurate only to order unity may however be $Q$-conserving, $E$-conserving or non-conserving. The $Q$-conserving forms yield zonal flow stability criteria but the others do not. In the other two problems studied, the $E$- and $Q$-conserving, order-unity approximations give no better behaviour than the non-conserving ones, and in some cases they actually give worse behaviour. No single order-unity, non-conserving model is uniformly better than the $E$- and $Q$-conserving forms, however; and it should be borne in mind that these conserving forms are amongst the most drastic approximations as regards the number of terms omitted from the governing QG1 equations. A clear, and expected, result is that order-$B$ approximations are in practice better than (or at least as good as) order unity approximations.

The $Q$- and $E$-conserving models which have order-$B$ formal accuracy retain some terms of order $B^2$. The results therefore provide support for the hypothesis that the practical accuracy of an approximate model may be increased by including higher-order terms in such a way that the conservation properties of the model are improved. The suggestion is strengthened by the fact that the order-$B$, $Q$-conserving model retains one term fewer than does the corresponding $E$-conserving model.

Various extensions of this study could be made in order to investigate its suggestions farther. It has been noted that some of the approximate long-wave models are consistently posed only for motion which is in a certain sense quasi-linear, and attention has been limited to such cases here (see (3.8)). This restriction may be relaxed by adopting a threefold decomposition of the streamfunction whereby the height average part $\psi_1$ is sub-divided into its zonal average and the departure from that average. Nine tracer coefficients may then be introduced and the development of §3 paralleled in every important respect. Details are given in White (1986*a*). An advantageous feature is that the class of order-$B$ models now includes non-conserving as well as $Q$-conserving and $E$-conserving forms. Thus it should be possible to use the threefold problem to discriminate between these different forms at order $B$, which is not possible in the twofold case (see above). However, test problems which can realize this possibility have not yet been devised. As might be expected, ambiguities arise when the problems considered in §§4.1 and 4.3 are tackled. For example, the finite-amplitude Rossby-wave problem leads to conditions involving only four of the nine $n_{ij}$, and it turns out that the behaviour deduced for any given $Q$-conserving model will also be exhibited by a number of $E$-conserving and non-conserving models. Nevertheless, it can be shown that only $Q$-conserving approximations give analogues of the QG1 stability criteria. Test problems which might give similarly unambiguous results include the following: the baroclinic instability of zonal flows having meridional as well as vertical shear; finite-amplitude motion forced by topography and diabatic heating (Derome 1984); the triad instability of baroclinic Rossby waves; and conditions for the existence of smooth solutions (Bennett & Kloeden 1981). Other extensions also seem worth pursuing. Amongst various refinements of the stream function decomposition method would be application to the case in which no rigid upper boundary is present, and to the non-Doppler quasi-geostrophic equations (White 1982). Finally, model 01 seems worthy of further investigations as a model of intermediate scale motion in the oceans (see §3.5), independent of the general approximation problem.

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